MODELLING OF STOCK PRICE CHANGES: A REAL ANALYSIS APPROACH

Rimas Norvaiša*

Institute of Mathematics and Informatics, Akademijos 4, LT-2600 Vilnius, Lithuania (e-mail: norvaisa@ktl.mii.lt)

Abstract. In this paper a real analysis approach to stock price modelling is considered. A stock price and its return are defined in a duality to each other provided there exist suitable limits along a sequence of nested partitions of a time interval, mimicking sum and product integrals. It extends the class of stochastic processes susceptible to theoretical analysis. Also, it is shown that extended classical calculus is applicable to market analysis whenever the local 2–variation of sample functions of the return is zero, or is determined by jumps if the process is discontinuous. In particular, an extended Riemann-Stieltjes integral is used in that case to prove several properties of trading strategies.

Key words: continuous–time model, model testing, stock price, return, trading strategy

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1 Introduction and discussion

In continuous–time financial mathematics the solution to the Doléans–Dade stochastic differential equation is often used as a model for stock price changes. The semimartingale driving this equation is called the return. Since many conclusions on the price behavior depend on the return, it plays an important role in mathematics of finance. On the other hand, the returns in econometrics of financial markets are sometimes modelled by stochastic processes which are not semimartingales. To provide a theoretical justification for such cases, one introduces a Doléans–Dade type equation with the stochastic integral replaced by a different integral. However, solutions to integral equations based on different integrals may differ considerably as demonstrated Wong and Zakai (1965). One may ask then whether it is possible to build up a model of stock price changes which is independent of a particular integration theory? The present paper addresses this question and provides a new insight into the relation between theoretical and applied financial mathematics.

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1.1. Prices and returns. To begin with we discuss two continuous–time stochastic models for a frictionless stock market. Let $R = \{R(t): 0 \le t \le T\}$ be a semimartingale such that $R(0) = 0$ almost surely and let $Q = \{Q(t): 0 \le t \le T\}$, where $Q(t) := \exp\{R(t)\}$ for $0 \le t \le T$. The pair (Q, R) will be called the *exponential system* of a stock. Then Q is the price and R is the return of a stock of the exponential system (Q, R). Let $P = \{P(t): 0 \le t \le T\}$ be a stochastic process satisfying the equation

$$P(t) = 1 + (SI) \int_0^t P(s-)\, dR(s), \quad 0 \le t \le T, \tag{1.1}$$

where $P(0-) := 1$ and (SI) denotes the stochastic integral defined by the L^2-isometry. Doléans–Dade (1970) proved that the unique solution to (1.1) is given by

$$P(t) = \exp\{R(t) - \frac{1}{2}\langle R^c, R^c\rangle(t)\} \prod_{(0,t]} (1 + \Delta R) \exp\{-\Delta R\}, \quad 0 < t \le T,$$

and $P(0) = 1$, where R^c is the continuous local martingale part of the semimartingale R and $\Delta R(s) := R(s) - R(s-)$ for $s \in (0, T]$. If P satisfies (1.1) and is bounded away from zero then, by associativity of the stochastic integral, we have

$$R(t) = (SI) \int_0^t \frac{dP(s)}{P(s-)}, \quad 0 \le t \le T. \tag{1.2}$$

The pair (P, R) satisfying (1.1) will be called the *stochastic exponential system.* Then P is the price and R is the return of a stock of the stochastic exponential system. Parts of continuous–time financial mathematics based on the exponential system and on the stochastic exponential system will be called respectively the exponential model and the stochastic exponential model. In general, the exponential system is different from the stochastic exponential system. Indeed, if R is a standard Brownian motion $B = \{B(t): t \ge 0\}$, then the solution to (1.1) is the stochastic process $P_B(t) := \exp\{B(t) - t/2\}$, $0 \le t \le T$, often called the geometric Brownian motion. In both systems the prices are observable quantities meaning that they represent real data, while the returns are non–observable and depend on the models. In addition to being a semimartingale, R may sometimes be assumed to satisfy certain probabilistic conditions about its distribution. An adequacy to real data of such assumptions on R can be tested by using the price transformations: the log return $R(t) = \log Q(t)$, $0 \le t \le T$, for the exponential model, and the return (1.2) for the stochastic exponential model. The log return is often used in econometric literature which means that certain hypotheses about the exponential model are tested. If one wishes to test the stochastic exponential model then the return (1.2) has to be used. However (1.2) is *not* defined for a single sample function, so that its statistical tractability is problematic. On the other hand, as pointed out Bühlmann, Delbaen, Embrechts and Shiryaev (1996), under probabilistic price analysis, the stochastic exponential model turns out to be more advantageous than the exponential model. Therefore, it is appealing to modify the stochastic exponential model in such a way that to make it more manageable for statistical analysis. Bühlmann et al. (1996) provide the analysis of the exponential model via its reduction to the stochastic exponential model using a suitable transformation in (1.1) instead of R.

1.2. Price changes as an evolution. In the present paper we define a price and its return in a duality without a priori integrability or probabilistic assumptions (Definition 2.9 below), and show that almost all sample functions of many typical stochastic processes including a Brownian motion satisfy the new definition (Propositions 2.10 and 2.11 below). An idea behind the definition is based on known results about a one–to–one correspondence between an evolution and its generator. A family of real numbers $U = \{U(s,t)\colon a \le s \le t \le b\}$ is an evolution on $[a,b]$ if $U(t,t) = 1$ and

$$U(t,r)U(r,s) = U(t,s) \quad \text{for all} \quad a \le s \le r \le t \le b.$$

Let P be a function representing a stock price over a period $[0,T]$ such that $P(0) = 1$, and let $U(t,s) := P(t)/P(s)$ for $0 \le s \le t \le T$. Then U so defined is a simple example of an evolution on $[0,T]$ defined by stock price changes. An evolution arise in describing the state of *nonautonomous* systems and they are generalizations of the concept of a one–parameter semigroup of bounded linear operators on a Banach space describing the state of autonomous linear systems. The classical Hille–Yosida theorem describes any strongly continuous, contractive semigroup in terms of its generator. In this way the Hille–Yosida theorem provides a one–to–one correspondence between semigroups and their generators. An important difficult question is when and in what sense will a given evolution U have a generator? The answer depends on a behaviour of the function $[a,b] \ni t \mapsto U(t,a)$, in particular on its p-variation. If U is defined by stock price changes, that is if $U(\cdot,0) = P$, then its generator is a return as defined in the present paper. Therefore the pair (P,R) satisfying Definition 2.9 is called the (*weak*) *evolutionary system*.

1.3. The p-variation. For the approach advocated in the present paper, the notion of p-variation of a function plays a role comparable with a role of a martingale property in the stochastic exponential model. For a function $f\colon [a,b] \mapsto \mathbb{R}$ and a real number $0 < p < \infty$, the *p-variation* $v_p(f) = v_p(f;[a,b])$ is the least upper bound of sums $s_p(f;\kappa) := \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|^p$ over all partitions $\kappa = \{x_i\colon i = 0,\dots,n\}$ of $[a,b]$. We notice that the 2-variation is *not* the same as the quadratic variation. For a standard Brownian motion $B = \{B(t)\colon t \ge 0\}$ and any $0 < T < \infty$, $v_2(B;[0,T]) = +\infty$ almost surely, while $v_p(B;[0,T]) < \infty$ for each $p > 2$ and the quadratic variation of B is defined in the almost sure sense for certain sequences of partitions. For any function f on $[a,b]$, define the *index of p-variation* $\upsilon(f) = \upsilon(f;[a,b])$ by

$$\upsilon(f;[a,b]) := \begin{cases} \inf\{p > 0\colon v_p(f) < \infty\} & \text{if the set is nonempty} \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore for a Brownian motion B, $\upsilon(B;[0,T]) = 2$ almost surely. Also for any $0 < T < \infty$, $\upsilon(X;[0,T]) < 2$ almost surely if X is a mean zero Gaussian stochastic process with stationary increments, continuous in quadratic mean and the incremental variance $\{E[X(t+u) - X(t)]^2\}^{1/2}$ varies regularly as $u \downarrow 0$ with index $\gamma > 1/2$, or if X is a homogeneous Lévy process with the Lévy measure L such that $\int_{\mathbb{R}\setminus\{0\}} (1 \wedge |x|^p)\, L(dx) < \infty$ for some $p < 2$.

1.4. Stochastic and classical calculi. In this paper it is proved that evolutionary systems (P,R) possess a uniformity property with respect to sequences of partitions defining P and R provided the p-variation index $\upsilon(R) < 2$. This fact is important

when one deals with fitting a model to real data, or when one considers a relation between discrete–time and continuous–time models (cf. Theorem 3.5 below). With the help of the result of Föllmer (1981) one can show that a weak evolutionary system (P, R) satisfies an integral equation similar to (1.1) with a different integral. If sample functions of the return R in the evolutionary system (P, R) have the p-variation index $\upsilon(R) < 2$, then (1.1) and (1.2) hold path by path with the stochastic integral replaced by the Left Young integral, an extended Riemann–Stieltjes integral. Norvaiša (1999) proved that the values of the Left Young integral and the values of the corresponding stochastic integral agree almost surely under conditions ensuring the existence of both. In this sense the value $\upsilon(R) = 2$ of the p-variation index is a borderline between an area where classical calculus applies and an area where stochastic calculus is needed essentially. We notice that the semimartingale property of the return R in the stochastic exponential system (P, R) makes a borderline between classical and stochastic calculi on a different level, e.g. the value $\upsilon(R) = 1$ (the 1-variation is the same as the total variation). Several examples of returns such as a hyperbolic Lévy motion, a normal inverse Gaussian Lévy process, the V.G. process, an α-stable Lévy motion with $\alpha \in [1, 2)$, or a fractional Brownian motion with the Hurst index $H \in (1/2, 1)$, can be treated using classical calculus. The aim of the present paper is to find a connection between the two calculi for the mathematics of finance. However, more interesting is a question whether it is possible to develop a full fledged model of a financial market based on the evolutionary system. Clearly it is not possible to answer to this question at this writing. A model construction requires a more advanced development of theories of integral equations and optimal control for functions of bounded p-variation, as well as further development of concepts of market efficiency, equilibrium and risk in the new context.

1.5. Arbitrage. We finish with a discussion of arbitrage for the evolutionary system. In the continuous–time financial mathematics based on the semimartingale theory the first and second fundamental theorems deal with the key principals of the theory. These theorems relate suitable forms of an arbitrage with the existence of a (unique) martingale measure and completeness. Thus an applicability of these tools is restricted if arbitrage is possible. In particular, this concerns the contingent claim valuation theory based on the no arbitrage principle. Less formally, the no arbitrage principle is considered as a natural property of a model of an *ideal* financial market because "there is no such thing as a free lunch" in equilibria market. These arguments may give an impression that no approaches other than martingale based stochastic calculus can be useful for mathematical finance. However the situation is not as simple as it may look. There are examples from a game theory where such a thing as a free lunch is possible under equilibria (see p. 137 in Kac, Rota and Schwartz, 1992). On the other hand, non–equilibrium can explain stylized facts discovered through the statistical analysis of market data (see Chapter 4 of an overview of Focardi and Jonas, 1997, based on interviews with over 100 persons in industry and academia). A strong critique of a whole current financial mathematics comes from actuaries who use different principles to value contingent claims (see e.g. Clarkson, 1996, 1997). So instead of avoiding arbitrage it seems more fruitful to have a model which accommodates both, free lunch areas as well as areas without a free lunch, and leave the question of performance evaluation of such a model to econometrics.

Next we illustrate how a real analysis approach may shed new light on arbitrage.

One way to define an arbitrage for evolutionary systems is to follow the pattern from the stochastic exponential model which requires first to define a self–financing strategy. As pointed out Harrison and Pliska (1981, Section 7), the restriction to predictable trading strategies as well as to gains defined using the stochastic integral needs a careful study. Clearly we cannot use these constructions in the present setting. Instead we define self–financing strategies pathwise following the logic of the present approach (Definition 3.3 below), and prove that the criteria suggested by Harrison and Pliska (1981) does apply to the new notions (Theorem 3.5 below). Then arbitrage can be defined either for a single function representing a price evolution, or for almost every sample function of a stochastic process using notation of Section 3 as follows: given a price $P=(P_0,\ldots,P_\nu)$ (of $1+\nu$ assets) during a time period $[0,T]$, a self–financing P–trading strategy $\phi=(\phi_0,\ldots,\phi_\nu)$ is an *arbitrage opportunity* for P at time T if the portfolio value function $V^{\phi,P}$ is 0 at 0 and positive at T. Salopek (1998) proved that an arbitrage in this sense can be constructed whenever the return of an evolutionary system is *continuous* function of bounded p-variation for some $1\le p<2$. To give a short proof of the same statement we modify the ingenious construction of an arbitrage due to Shiryaev (1998, Example VII.2c.4). To this aim we replace a fractional Brownian motion with the Hurst index $H\in(1/2,1)$ in his construction, with a continuous function of bounded p-variation for some $1\le p<2$, and use the chain rule formula given by Theorem 2.1 below instead of Itô's formula.

Proposition 1.1. *Let f be a continuous function of bounded p-variation on $[0,T]$ for some $1\le p<2$ such that $f(0)=0$ and $f(T)\neq 0$, and let r, σ be real numbers. Then for the evolutionary system (P,R) with $R(t)=(rt,rt+\sigma f(t))$, $0\le t\le T$, there exists an arbitrage opportunity for P at time T.*

Proof. Let $R_0(t):=rt$ and $R_1(t):=rt+\sigma f(t)$ for $0\le t\le T$. By Proposition 2.6, $P_0(t)=e^{rt}$ and $P_1(t)=e^{rt+\sigma f(t)}$ for $0\le t\le T$. The vector function $P=(P_0,P_1)$ is the price in the sense defined in Section 3 below. Let $\phi_0(t):=1-\exp\{2\sigma f(t)\}$ and $\phi_1(t):=2[\exp\{\sigma f(t)\}-1]$ for $0\le t\le T$. By Proposition 3.2, $\phi=(\phi_0,\phi_1)$ is the P–trading strategy. Next we show that ϕ is self–financing P–trading strategy as defined in Definition 3.3. To this aim we apply the chain rule formula from Theorem 2.1 twice: first take $h\equiv 1$, $F(u_1,u_2)=u_1u_2^2$, and second take $h=\phi_1$, $F(u_1,u_2)=u_1u_2$. For each $0<t\le T$, we then have

$$
\begin{aligned}
V^{\phi}(t)=e^{rt}\Big[e^{\sigma f(t)}-1\Big]^2&=\int_0^t\Big[e^{\sigma f(t)}-1\Big]^2\,de^{rt}+\int_0^t 2e^{rt}\Big[e^{\sigma f(t)}-1\Big]\,de^{\sigma f(t)}\\
&=\int_0^t\Big[1-e^{2\sigma f(t)}\Big]\,de^{rt}+\int_0^t 2\Big[e^{\sigma f(t)}-1\Big]\,de^{rt+\sigma f(t)}=V^{\phi}(0)+G^{\phi}(t),
\end{aligned}
$$

where all integrals exist in the Riemann-Stieltjes sense by the Stieltjes integrability theorem of L.C. Young (1936, p. 264). Since $V^{\phi}(0)=0$ and $V^{\phi}(T)>0$, the self–financing P–trading strategy ϕ is an arbitrage opportunity for P at time T. $\square$

The preceding fact shows the irrelevance of a "long memory" of a fractional Brownian motion with respect to an arbitrage. Sample function behavior of a stochastic process is responsible for arbitrage opportunities. A Weierstrass function is a non–probabilistic example of a function f satisfying hypotheses of Proposition 1.1. Once we except the evolutionary system as a base for a model of a financial market then an

arbitrage is a property of a local behavior of a sample function rather than a correlation property between indefinitely increasing time moments.

2 Duality between price and return

2.1. Returns in discrete–time models. If time t is discrete, say $t = 0, 1, \ldots, T$, there are at least two different notions of return. Let $P = \{P(t)\colon t = 0, 1, \ldots, T\}$ be a price of a stock which pays no dividends. The *simple net return* $R_1 = \{R_1(t)\colon t = 0, 1, \ldots, T\}$ is defined by setting $R_1(0) := 0$, and for each $t = 1, \ldots, T$,

$$\widehat{R}_1(t) := R_1(t) - R_1(t-1) := \begin{cases} [P(t) - P(t-1)]/P(t-1), & \text{if } P(t-1) > 0, \\ 0, & \text{if } P(t-1) = 0. \end{cases} \tag{2.1}$$

Notice that $\widehat{R}_1(t)$ depends on values $P(t-1)$ and $P(t)$, so that $\widehat{R}_1$ is the function of a subinterval $[t-1, t]$. Notation $\widehat{R}_1$ (as well as $\widehat{R}_2$ defined below) is natural to use in discrete–time models where time lags have *fixed* length. A work with continuous–time models requires to treat returns either as interval functions defined on *all* subintervals of $[0, T]$, or as point functions on $[0, T]$. In this paper we choose to use the form of a point function. Given $P(0) > 0$, there is a one–to–one correspondence between a positive price P and a simple net return R_1 having jumps bigger than minus one, as described by Pliska (1997, Section 3.2). Namely, in addition to (2.1), for each $t = 1, \ldots, T$, we have

$$P(t) = P(0) + \sum_{s=1}^{t} P(s-1)\widehat{R}_1(s) \quad \text{and} \quad P(t) = P(0)\prod_{s=1}^{t}[1 + \widehat{R}_1(s)]. \tag{2.2}$$

This correspondence is used in security market models by specifying simple net returns rather than prices.

Another type of a return is the *log return* $R_2 = \{R_2(t)\colon t = 0, 1, \ldots, T\}$ defined by setting $R_2(0) := 0$ and, for all $t = 1, \ldots, T$,

$$\widehat{R}_2(t) := R_2(t) - R_2(t-1) := \begin{cases} \log[P(t)/P(t-1)], & \text{if } P(t-1) > 0, \\ 0, & \text{if } P(t-1) = 0. \end{cases}$$

This return is often used in the econometrics literature on security markets. It is easy to see that R_2 satisfies the additivity property

$$\widehat{R}_2(t) + \widehat{R}_2(t-1) + \cdots + \widehat{R}_2(t-s+1) = \log[P(t)/P(t-s)] \tag{2.3}$$

for any $s, t \in \{0, 1, \ldots, T\}$, $s < t$, because the right side is the log return for the time period between $t - s$ and t. The additivity property of log returns is one reason of its popularity among econometricians. For a discussion of these and other related properties of returns, see Campbell, Lo and MacKinlay (1997, Section 1.4.1).

Sometimes statistical conclusions based on the log return R_2 are applied to the model (2.2) or even to the continuous–time stochastic exponential model. To justify this one usually argues that $\widehat{R}_1$ and $\widehat{R}_2$ are relatively close to each other when both are small. However, there are cases when the difference between $\widehat{R}_1$ and $\widehat{R}_2$ cannot

be neglected (see e.g. Elton, Gruber and Kleindorfer, 1975). To test the stochastic exponential model one needs to use the return R defined via the Itô integral (1.2). Then one has to be able to evaluate the Itô integral using finitely many values of a single sample function. Given a sequence of partitions into shrinking subintervals of $[0,T]$, by the dominated convergence in probability theorem, the value of R can be approximated in probability by corresponding Riemann–Stieltjes sums. However, it is not possible to conclude the convergence with probability 1 without further restrictions. A relationship between the price and its return is suggested below for the continuous–time framework which makes this approximation possible path by path. This relationship is motivated by a duality relation between additive and multiplicative interval functions, which is also known as the evolution representation problem. Recall that the additivity property is satisfied by the log return R_2 (see (2.3)) while the simple net return R_1 lacks this property. On the other hand, the multiplicativity property

$$\widehat{P}(t)\cdot\widehat{P}(t-1)\cdot\ldots\cdot\widehat{P}(t-s+1)=P(t)/P(t-s)$$

for any $s,t\in\{0,1,\ldots,T\}$, $s<t$, is satisfied by the price ratios $\widehat{P}(t):=P(t)/P(t-1)$, $t=1,\ldots,T$.

2.2. The chain rule formula. For a finite interval J, open or closed at either end, let $Q(J)$ be the set of all partitions $\kappa=\{x_i\colon i=0,\ldots,n\}$ of J. As before for $f\colon J\mapsto\mathbb{R}$ and $0<p<\infty$, let $v_p(f):=v_p(f;J):=\sup\{s_p(f;\kappa)\colon \kappa\in Q(J)\}$ be the p-variation of f, where $s_p(f;\kappa):=\sum_{j=1}^{n}|f(x_i)-f(x_{i-1})|^p$ for $\kappa=\{x_i\colon i=0,\ldots,n\}$. Denote by $\mathcal{W}_p=\mathcal{W}_p(J)$ the set of all functions f such that $v_p(f)<\infty$. If $f\in\mathcal{W}_p$ for some $p<\infty$ then f is *regulated*, that is there exist the limits $f(x-):=\lim_{y\uparrow x}f(y)$ and $f(x+):=\lim_{y\downarrow x}f(y)$ when these are defined. The class of all regulated functions on J will be denoted by $\mathcal{R}(J)$. Given a regulated function f on $[a,b]$, define a left-continuous function $f_-^{(a)}$ and a right-continuous function $f_+^{(b)}$ by

$$\begin{cases} f_-^{(a)}(x):=f_-(x):=f(x-) & \text{for}\quad a<x\le b\quad\text{and}\quad f_-^{(a)}(a):=f(a)\\ f_+^{(b)}(x):=f_+(x):=f(x+) & \text{for}\quad a\le x<b\quad\text{and}\quad f_+^{(b)}(b):=f(b).\end{cases}$$

Given a regulated function f on J, define Δ^-f on J by $\Delta^-f(x):=f(x)-f(x-)$ if $f(x-)$ is defined and $\Delta^-f(x):=0$ otherwise. Similarly define Δ^+f on J by $\Delta^+f(x):=f(x+)-f(x)$ if $f(x+)$ is defined and $\Delta^+f(x):=0$ otherwise. Since each regulated function f has at most countably many jumps, one can define $\mathfrak{S}_p(f):=\mathfrak{S}_p(f;J):=\{\sum_J(|\Delta^-f|^p+|\Delta^+f|^p)\}^{1/p}$. The *local p-variation* $v_p(f)^*:=v_p(f;J)^*$ is defined by

$$v_p(f;J)^*:=\inf_{\lambda\in Q(J)}\sup\{s_p(f;\kappa)\colon \lambda\subset\kappa\in Q(J)\}.$$

Then we have the relation $\mathfrak{S}_p(f)^p\le v_p^*(f)\le v_p(f)$. Let $\mathcal{W}_p^*=\mathcal{W}_p^*(J):=\{f\in\mathcal{W}_p\colon \mathfrak{S}_p(f)^p=v_p^*(f)\}$ for $1<p<\infty$. For regulated functions h and f on $[a,b]$, define the *Left Young integral*, or the *LY* integral, by

$$(LY)\int_a^b h\,df:=(RS)\int_a^b h_-^{(a)}\,df_+^{(b)}+\big[h\Delta^+f\big](a)+\sum_{(a,b)}\Delta^-h\Delta^+f$$

provided the Riemann–Stieltjes integral exists in the refinement sense and the sum converges absolutely. Additivity on adjacent intervals as well as some other properties of the LY integral are proved in Norvaiša (1999). From the Stieltjes integrability theorem of L.C. Young (1936) it follows that $(LY)\int_a^b h\,df$ is defined if $h \in \mathcal{W}_p$, $f \in \mathcal{W}_q$ and $1/p + 1/q > 1$. The following theorem of Norvaiša (1999) extends this result to the case when $1/p + 1/q = 1$ under additional assumptions on h and f. Let ν be a positive integer, and let F be a real-valued function defined on an open set $U \subset \mathbb{R}^\nu$ containing a ν-dimensional cube $[c,d]^\nu := [c,d] \times \cdots \times [c,d]$. We write $F \in \Lambda_{1,\alpha}([c,d]^\nu)$ for $\alpha \in (0,1]$ if F is differentiable on U with partial derivatives F'_l, $l = 1, \ldots, \nu$, and there is a finite constant K_α such that the inequality

$$\max_{1 \le l \le \nu} |F'_l(u) - F'_l(v)| \le K_\alpha \sum_{k=1}^{\nu} |u_k - v_k|^\alpha$$

holds for all $u = (u_1, \ldots, u_\nu), v = (v_1, \ldots, v_\nu) \in [c,d]^\nu$.

Theorem 2.1. *For $\alpha \in (0,1]$, let $f = (f_1, \ldots, f_\nu)\colon [a,b] \mapsto (c,d)^\nu$ be a vector function with coordinate functions $f_l \in \mathcal{W}^*_{1+\alpha}([a,b])$ for $l = 1, \ldots, \nu$, let $F \in \Lambda_{1,\alpha}([c,d]^\nu)$ and let h be a regulated function on $[a,b]$. Then the equality*

$$(LY)\int_a^b h\,d(F\circ f) = \sum_{l=1}^{\nu} (LY)\int_a^b h(F'_l\circ f)\,df_l$$

$$+\sum_{(a,b]} h_-\big[\Delta^-(F\circ f) - \sum_{l=1}^{\nu}(F'_l\circ f)_-\Delta^- f_l\big] + \sum_{[a,b)} h\big[\Delta^+(F\circ f) - \sum_{l=1}^{\nu}(F'_l\circ f)\Delta^+ f_l\big]$$

holds meaning that all $\nu+1$ integrals exist provided any d integrals exist, and the two sums converge absolutely.

We refer to the preceding statement as the chain rule formula. Its proof is given by Norvaiša (1999).

2.3. Duality relation. Turning to a continuous–time model, consider an interval $[0,T]$, $0 < T < \infty$. Roughly speaking, to extend (2.1) and (2.2) to functions defined on $[0,T]$, we pass to a limit along a nested sequence λ of partitions of $[0,T]$. The first limit $\mathcal{L}_\lambda(f)$ if exists is an extension of the (sum) integral, and for $f \in \mathcal{W}^*_2$, its values coincide with values of the LY integral. The second limit $\mathcal{E}_\lambda(g)$ if exists is an extension of the product integral, and for sample functions g of a semimartingale, its values coincide with values of the solution to the Doléans-Dade equation (1.1). First we prove the existence of $\mathcal{L}_\lambda(f)$ and $\mathcal{E}_\lambda(g)$ for functions f and g from the class $\mathcal{W}^*_2$. Then the duality relations (2.13) are derived for such functions. Finally, a duality relation is proved for functions having defined the quadratic variation.

Definition 2.2. Let $\mathfrak{Q}([0,T])$ be the set of all nested sequences $\lambda = \{\lambda(m)\colon m \ge 1\}$ of partitions $\lambda(m) = \{0 = t_0^m < \cdots < t_{n(m)}^m = T\}$ of $[0,T]$ such that $\cup_m \lambda(m)$ is dense in $[0,T]$. Let I_T be either $[0,T]$ or $[0,T)$, and let f be a real–valued function on I_T. Given $\lambda \in \mathfrak{Q}([0,T])$, we say that $\mathcal{L}_\lambda = \mathcal{L}_\lambda(f)$ *is defined* on I_T if the limit

$$\mathcal{L}_\lambda(f)(t) := \lim_{m\to\infty} \sum_{i=1}^{n(m)} [f(t_i^m \wedge t) - f(t_{i-1}^m \wedge t)]/f(t_{i-1}^m \wedge t) \tag{2.4}$$

exists for each $t \in I_T$. Given a nonempty subset $\mathfrak{Q} \subset \mathfrak{Q}([0,T])$, if $\mathcal{L}_\lambda(f)$ is defined for and does not depend on each $\lambda \in \mathfrak{Q}$, then set $\mathcal{L}_{\mathfrak{Q}} = \mathcal{L}_{\mathfrak{Q}}(f)$ to be equal to any $\mathcal{L}_\lambda(f)$, $\lambda \in \mathfrak{Q}$.

For a regulated function f, a typical example of $\mathfrak{Q} \subset \mathfrak{Q}([0,T])$ in the preceding definition is the set $\mathfrak{Q}(f)$ defined by

$$\mathfrak{Q}(f) := \begin{cases} \mathfrak{Q}([0,T]), & \text{if } f \in \mathcal{D}(I_T), \\ \{\lambda \in \mathfrak{Q}([0,T])\colon \cup_m \lambda(m) \supset \Delta_f(I_T)\}, & \text{if } f \in \mathcal{R}(I_T) \setminus \mathcal{D}(I_T), \end{cases} \tag{2.5}$$

where $f \in \mathcal{D}(I_T)$ if, at each point of $(0,T)$, f is either right-continuous or left-continuous and $\Delta_f(I_T) := \{x \in (0,T)\colon \Delta^- f(x) \neq 0 \text{ or } \Delta^+ f(x) \neq 0\}$. Next, under stated conditions we show that $\mathcal{L}_{\mathfrak{Q}(f)}$ is defined and has values of the indefinite LY integral.

Proposition 2.3. *Let $f \in \mathcal{W}_2^*(I_T)$ and let $\inf\{f(t)\colon t \in I_T\} \geq \delta$ for some $\delta > 0$. Then $\mathcal{L}_{\mathfrak{Q}(f)}(f)$ is defined on I_T. Moreover, for each $t \in I_T$, f^{-1} is LY integrable with respect to f on $[0,t]$ and the relation*

$$\mathcal{L}_{\mathfrak{Q}(f)}(f)(t) = (LY)\int_0^t \frac{df}{f} = \log\frac{f(t)}{f(0)} - \sum_{(0,t]}\left[\log\frac{f}{f_-} - \frac{\Delta^- f}{f_-}\right] - \sum_{[0,t)}\left[\log\frac{f_+}{f} - \frac{\Delta^+ f}{f}\right] \tag{2.6}$$

holds, where the two sums converge absolutely.

For the proof we need an auxiliary statement, where $Q(S) := \{\kappa \in Q(J)\colon \kappa \subset S\}$ for any subset $S \subset J$.

Lemma 2.4. *Let $f \in \mathcal{W}_p^*([a,b])$ for some $1 < p < \infty$, and let S be a dense subset of $[a,b]$ containing all discontinuity points of f. For each $\epsilon > 0$, there exists $\lambda \in Q(S)$ such that $\sum_{j=1}^k v_p(f;(z_{j-1},z_j)) < \epsilon$ whenever $\lambda \subset \{z_j\colon j = 0,\dots,k\} \in Q([a,b])$.*

Proof. Let $S \subset [a,b]$ be as in the statement. Then $v_p^*(f;[a,b])$ is equal to the greatest lower bound of sums $\sum_{j=1}^k v_p(f;[z_{j-1},z_j])$ over $\{z_j\colon j = 0,\dots,k\} \in Q(S)$. Let $\epsilon > 0$. Since $\mathfrak{S}_p(f) < \infty$, there exists a finite set $\mu \subset [a,b]$ such that $\sum_\nu(|\Delta^- f|^p + |\Delta^+ f|^p) > \mathfrak{S}_p(f)^p - \epsilon/2$ for each $\nu \supset \mu$. Then one can choose $\lambda \in Q(S)$ such that $\lambda \supset \mu$ and $\sum_{j=1}^k v_p(f;[z_{j-1},z_j]) < v_p^*(f) + \epsilon/2$ whenever $\lambda \subset \{z_j\colon j = 0,\dots,k\} \in Q([a,b])$. For each small enough $\delta > 0$ and each $j = 1,\dots,k$, we have $v_p(f;[z_{j-1},z_j]) \geq v_p(f;[z_{j-1},z_{j-1}+\delta]) + v_p(f;[z_{j-1}+\delta,z_j-\delta]) + v_p(f;[z_j-\delta,z_j])$. Letting $\delta \downarrow 0$ and using Lemma 2.19 of Dudley and Norvaiša (1999), we get

$$\begin{aligned}\sum_{j=1}^k v_p(f;(z_{j-1},z_j)) &\leq \sum_{j=1}^k v_p(f;[z_{j-1},z_j]) - \sum_{j=1}^k \left[|\Delta^- f(z_j)|^p + |\Delta^+ f(z_{j-1})|^p\right] \\ &< V_p^*(f)^p + \epsilon/2 - \mathfrak{S}_p(f)^p + \epsilon/2 = \epsilon.\end{aligned}$$

The proof of Lemma 2.4 is complete. $\square$

Proof of Proposition 2.3. The existence of the integral $(LY)\int_0^t f^{-1}\,df$ and the second equality in (2.6) for each $t \in I_T$ follow from Theorem 2.1. To see if it's true take

$F(u) := \log u$ for $u \in [\delta, \|f\|_\infty]$, $\nu = 1$, $\alpha = 1$, $h \equiv 1$, and notice that $\Delta^-(F\circ f) = \log(f/f_-)$, $\Delta^+(F\circ f) = \log(f_+/f)$. To prove that $\mathcal{L}_{\mathfrak{Q}(f)}(f)$ is defined on I_T and that the first equality in (2.6) holds, for each $u \in (0,T] \cap I_T$ and $v \in [0,T)$, let

$$\phi^-(u) := \log \frac{f(u)}{f(u-)} - \frac{\Delta^- f(u)}{f(u-)} \quad \text{and} \quad \phi^+(v) := \log \frac{f(v+)}{f(v)} - \frac{\Delta^+ f(v)}{f(v)}.$$

To begin with the second case in (2.5) consider $\{\lambda(m): m \geq 1\} \in \mathfrak{Q}([0,T])$ such that $\cup_m \lambda(m)$ contains all discontinuity points of f. For each $m \geq 1$, let

$$\psi_m(i) := \log \frac{f(t_i^m)}{f(t_{i-1}^m)} - \Big[\frac{f(t_i^m)}{f(t_{i-1}^m)} - 1\Big] \qquad \text{for } i = 1, \ldots, n(m).$$

Let $S := \cup_m \lambda(m)$ and $\epsilon > 0$. By Lemma 2.4, and because f is regulated and the two sums in (2.6) converge absolutely, one can choose $\kappa := \{z_j: j = 0, \ldots, k\} \in Q(S)$ such that

$$\sum_{j=1}^k v_2(f; (z_{j-1}, z_j)) < \epsilon, \quad \max_{1\leq j\leq k} Osc(f; (z_{j-1}, z_j)) < \frac{\delta}{2} \quad \text{and} \quad \sum_\nu \big(|\phi^-| + |\phi^+|\big) < \epsilon$$

for any $\nu \subset I_T \setminus \kappa$. For each partition $\lambda(m) = \{t_i^m: i = 0, \ldots, n(m)\}$ containing κ and for each $j \in \{0, \ldots, k\}$, let $i(j) \in \{0, \ldots, n(m)\}$ be an index such that $z_j = t_{i(j)}^m$. Since

$$\lim_{m\to\infty} \psi_m(i(j)) = \phi^-(z_j) \quad \text{and} \quad \lim_{m\to\infty} \psi_m(i(j-1)+1) = \phi^+(z_{j-1}) \tag{2.7}$$

for $j = 1, \ldots, k$, one can choose an integer $M \geq 1$ such that $\lambda(M) \supset \kappa$, there are at least two elements of $\lambda(M)$ in each interval $(t_{i(j-1)}^m, t_{i(j)}^m)$, $j = 1, \ldots, k$, and

$$\Big| \sum_{j=1}^k \Big[\psi_m(i(j)) - \phi^-(z_j)\Big] + \sum_{j=0}^{k-1} \Big[\psi_m(i(j)+1) - \phi^+(z_j)\Big] \Big| < \epsilon$$

for $m \geq M$. Let $t \in I_T$. First suppose $t \in S$, so that $t = t_{l(m)}^m$ for some $l(m) \in \{1, \ldots, n(m)\}$ and for all m larger than some $N(t)$. For each $m \geq M \vee N(t)$, let $l := \max\{j \leq k: z_j \leq t\}$ and $J := \{0, \ldots, l(m)\} \setminus \{i(j), i(j-1)+1: j = 1, \ldots, l\}$. By the Taylor series expansion with remainder, we have $|\log(1+u) - u| \leq 2u^2$ for each $|u| \leq 1/2$. Then, for all $m \geq M \vee N(t)$, we get

$$\Big| \log \frac{f(t)}{f(0)} - \sum_{(0,t]} \phi^- - \sum_{[0,t)} \phi^+ - \sum_{i=1}^{n(m)} \big[\frac{f(t_i^m \wedge t)}{f(t_{i-1}^m \wedge t)} - 1\big] \Big|$$

$$< 2\epsilon + 2 \sum_{i\in J} \Big| \frac{f(t_i^m)}{f(t_{i-1}^m)} - 1 \Big|^2 < 2\epsilon + \frac{2}{\delta^2} \sum_{j=1}^k v_2(f; (z_{j-1}, z_j)) < 2\epsilon(1 + \delta^{-2}). \tag{2.8}$$

If $t \in I_T \setminus S$ then we have in addition the term

$$\Big| \log \frac{f(t)}{f(t_{l(m)}^m)} - \Big[\frac{f(t)}{f(t_{l(m)}^m)} - 1\Big] - \sum_{(t_{l(m)}^m, t]} \phi^- - \sum_{[t_{l(m)}^m, t)} \phi^+ \Big|,$$

where $l(m) := \max\{i \le n(m)\colon t_i^m < t\}$. This term tends to zero as $m \to \infty$ because f is continuous at t in this case. Since ϵ in (2.8) is arbitrary, $\mathcal{L}_{\mathfrak{Q}(f)}(f)$ is defined on I_T and the first equality in (2.6) holds for the second case in (2.5). The proof when $f \in \mathcal{D}(I_T)$ is the same except that we use Lemma 2.4 with $S = I_T$, and choose $\{t_i(j)^m\colon j = 0, \dots, k\}$ so that $z_j \in (t^m_{i(j)-1}, t^m_{i(j)}]$ if f is right–continuous at z_j and $z_j \in [t^m_{i(j)}, t^m_{i(j)+1})$ if f is left–continuous at z_j. The proof of Proposition 2.3 is complete. $\square$

Using notation as in Definition 2.2, we have:

Definition 2.5. Let g be a real–valued function on I_T. Given $\lambda \in \mathfrak{Q}([0,T])$, we say that $\mathcal{E}_\lambda = \mathcal{E}_\lambda(g)$ *is defined* on I_T if the limit

$$\mathcal{E}_\lambda(g)(t) := \lim_{m\to\infty} \prod_{i=1}^{n(m)} [1 + g(t_i^m \wedge t) - g(t_{i-1}^m \wedge t)] \tag{2.9}$$

exists for each $t \in I_T$. Given a nonempty subset $\mathfrak{Q} \subset \mathfrak{Q}([0,T])$, if $\mathcal{E}_\lambda(g)$ is defined for and does not depend on each $\lambda \in \mathfrak{Q}$, then we define $\mathcal{E}_{\mathfrak{Q}} = \mathcal{E}_{\mathfrak{Q}}(g)$ to be equal to any $\mathcal{E}_\lambda(g)$, $\lambda \in \mathfrak{Q}$.

Next, under stated conditions we show that (2.9) is defined and has values of the *product integral with respect to g over* $[0,t]$, $\prod_0^t(1+dg)$, defined as the limit of the product from $i = 1$ to n of $1 + g(t_i) - g(t_{i-1})$, if it exists, under refinements of partitions $\{t_i\colon i = 0, \dots, n\}$ of $[0,t]$. The set $\mathfrak{Q}(g)$ in the following statement is defined by (2.5).

Proposition 2.6. *Let $g \in \mathcal{W}_2^*(I_T)$ and let $(\Delta^- g)\wedge(\Delta^+ g) > -1$ on I_T. Then $\mathcal{E}_{\mathfrak{Q}(g)}(g)$ is defined on I_T. Moreover, for each $t \in I_T$, the product integral $\prod_0^t(1+dg)$ exists, is positive and the relation*

$$\mathcal{E}_{\mathfrak{Q}(g)}(g)(t) = \prod_0^t (1+dg) = e^{g(t)-g(0)} \prod_{[0,t]} \big[(1+\Delta^- g)(1+\Delta^+ g)\big] e^{-\Delta^- g - \Delta^+ g} \tag{2.10}$$

holds, where the product converges absolutely.

Proof. The product integral $\prod_0^t(1+dg)$ exists, and the second equality in (2.10) holds for each $t \in I_T$ by Theorem 4.4 of Dudley and Norvaiša (1999). Since all jumps of g are bigger than -1, the positivity of the product integral follows from its definition. To prove that $\mathcal{E}_{\mathfrak{Q}(g)}(g)$ is defined on I_T and the first equality in (2.10) holds, let $t \in I_T$ and $\{\lambda(m)\colon m \ge 1\} \in \mathfrak{Q}(g)$, where $\lambda(m) = \{t_i^m\colon i = 0, \dots, n(m)\}$. One can assume that $t = t^m_{l(m)}$ for some $1 \le l(m) \le n(m)$. Otherwise we include t into $\lambda(m)$ and change indices. For a finite set $\mu \subset I_T$, let

$$A(g;\mu) := \prod_{z\in\mu} \big[(1+\Delta^- g(z))(1+\Delta^+ g(z)\big] e^{-\Delta^- g(z) - \Delta^+ g(z)}.$$

To begin with the second case in (2.5) consider $\{\lambda(m)\colon m \ge 1\} \in \mathfrak{Q}([0,T])$ such that $\cup_m \lambda(m)$ contains all discontinuity points of g. Let $S := \cup_m \bar\lambda(m)$, where $\bar\lambda(m) = \{0 = t_0^m < \cdots < t^m_{l(m)} = t\}$, and $\epsilon \in (0, 2A)$, where $A(g)$ denotes the product $\prod_{[0,t]}$ in (2.10).

Because g is regulated and by Lemma 2.4, one can choose $\kappa = \{z_j\colon j = 0, \dots, k\} \in Q(S)$ such that $Osc(g(z_{j-1}, z_j)) < 1/2$ for $j = 1, \dots, k$,

$$\sum_{j=1}^{k} v_2(g; (z_{j-1}, z_j)) < \frac{\epsilon}{8eA} \quad \text{and} \quad \big|A(g;\mu) - A(g)\big| < \frac{\epsilon}{4}$$

for all $\mu \supset \kappa$. For each $\bar\lambda(m) \supset \kappa$ and for each $j \in \{0, \dots, k\}$, let $i(j) \in \{0, \dots, l(m)\}$ be such that $z_j = t^m_{i(j)}$. Let $\Delta^m_i g := g(t^m_i) - g(t^m_{i-1})$ for $i = 1, \dots, l(m)$, $m \geq 1$, and let

$$U(g;\kappa) := \prod_{j=1}^{k} (1 + \Delta^m_{i(j)} g)(1 + \Delta^m_{i(j-1)+1} g) \exp\big\{ -\Delta^m_{i(j)} g - \Delta^m_{i(j-1)+1} g \big\}.$$

Then letting $J := \{0, \dots, l(m)\} \setminus \{i(j), i(j-1)+1\colon j = 1, \dots, k\}$, we have

$$\prod_{i=1}^{l(m)} (1 + \Delta^m_i g) = e^{g(t)-g(0)} U(g;\kappa) \exp\Big\{ \sum_{i\in J} \big[\log(1 + \Delta^m_i g) - \Delta^m_i g \big] \Big\}, \tag{2.11}$$

for all $\bar\lambda(m) \supset \kappa$. Since $|\Delta^m_i g| \leq 1/2$ for $i \in J$, by the Taylor series expansion with remainder, we get

$$\Big| \sum_{i\in J} \big[\log(1 + \Delta^m_i g) - \Delta^m_i g \big] \Big| = \sum_{i\in J} \theta(\Delta^m_i g)(\Delta^m_i g)^2 \leq 2 \sum_{j=1}^{k} v_2(g; (z_{j-1}, z_j)) < \frac{\epsilon}{4eA},$$

where $\theta(u) \in [2/9, 2]$ for $|u| \leq 1/2$. Let $M \geq 1$ be an integer such that $\bar\lambda(M) \supset \kappa$, there are at least two elements of $\bar\lambda(M)$ in each interval $(t^m_{i(j-1)}, t^m_{i(j)})$, $j = 1, \dots, k$, and $|1 - U(g;\kappa)/A(g;\kappa)| < \epsilon/(8eA)$ for all $m \geq M$. Using the inequality $|ue^v - 1| \leq e|v| + 2e|u-1|$ for $|v| \leq 1/2$ and $|1-u| \leq 1/4$ one can show that (2.11) differs from the right side of (2.10) by $\epsilon \exp\{g(t) - g(0)\}$ for all $m \geq M$. Since ϵ is arbitrary, $\mathcal{E}_{\mathfrak{Q}(g)}(g)$ is defined on I_T and the first relation in (2.10) holds when $\mathfrak{Q}(g)$ is defined by the second case in (2.5). The proof for the first case of (2.5) is similar and therefore is omitted. The proof of Proposition 2.6 is complete. $\Box$

To show a duality between $\mathcal{E}_{\mathfrak{Q}}(g)$ and $\mathcal{L}_{\mathfrak{Q}}(f)$ for $g, f \in \mathcal{W}^*_2$ first we prove it between the *indefinite product integral* $\mathcal{P}(g)$ and the *indefinite LY integral* $\mathcal{S}(f)$ defined by

$$\mathcal{P}(g)(t) := \prod_0^t (1 + dg) \quad \text{and} \quad \mathcal{S}(f)(t) := (LY)\int_0^t \frac{df}{f} \tag{2.12}$$

for $t \in I_T$ whenever the integrals exist. The following theorem was proved by Dudley and Norvaiša (1999, Theorem 6.8 and 6.10) for functions f, g with values in a Banach algebra under the stronger assumption: $f, g \in \mathcal{W}_p$ for some $p \in (0, 2)$. To extend this result to real–valued functions from the class $\mathcal{W}^*_2$ we use the chain rule formula.

Theorem 2.7. I. *Let $g \in \mathcal{W}_2^*(I_T)$ and $(\Delta^- g)\wedge(\Delta^+ g) > -1$ on I_T. Then the indefinite product integral $\mathcal{P}(g)$ is defined and $\mathcal{P}(g) \in \mathcal{W}_2^*(I_T)$. Moreover, the indefinite LY integral in (2.12) is defined for $f = \mathcal{P}(g)$ and $\mathcal{S}(\mathcal{P}(g))(t) = g(t) - g(0)$ for $t \in I_T$.*

II. *Let $f \in \mathcal{W}_2^*(I_T)$ and $\inf\{f(t)\colon t \in I_T\} \geq \delta$ for some $\delta > 0$. Then the indefinite LY integral $\mathcal{S}(f)$ is defined and $\mathcal{S}(f) \in \mathcal{W}_2^*(I_T)$. Moreover, the product integral in (2.12) is defined for $g = \mathcal{S}(f)$ and $\mathcal{P}(\mathcal{S}(f))(t) = f(t)/f(0)$ for $t \in I_T$.*

Proof. I. The indefinite product integral $\mathcal{P}(g)$ is defined on I_T by Theorem 4.4 of Dudley and Norvaiša (1999). It is easy to prove that $\mathcal{P}(g)/\mathcal{P}(g)_- = 1 + \Delta^- g > 0$, $\mathcal{P}(g)_+/\mathcal{P}(g) = 1 + \Delta^+ g > 0$ on I_T, and $\mathcal{P}(g) \in \mathcal{W}_2^*(I_T)$. Let $t \in I_T$. By Theorem 2.1, the LY integral $\mathcal{S}(\mathcal{P}(g))(t)$ exists and

$$\mathcal{S}(\mathcal{P}(g))(t) = g(t) - g(0) + \log\Big[\prod_{(0,t]}(1+\Delta^- g)e^{-\Delta^- g}\Big] + \log\Big[\prod_{[0,t)}(1+\Delta^+ g)e^{-\Delta^+ g}\Big]$$

$$-\sum_{(0,t]}\big[\log(1+\Delta^- g) - \Delta^- g\big] - \sum_{[0,t)}\big[\log(1+\Delta^+ g) - \Delta^+ g\big] = g(t) - g(0).$$

The last equality follows by taking the limit of $\log[\prod_\mu \Phi] = \sum_\mu[\log\Phi]$ along a nested sequence of finite sets μ of jump points of $\Phi = (1+\Delta g)\exp(-\Delta g)$.

II. By Theorem 2.1, $\mathcal{S}(f)$ is defined on I_T and its value is given by the right side of (2.6). Since $\log f \in \mathcal{W}_2^*$ and the two sums in (2.6) converge absolutely, it follows that $\mathcal{S}(f) \in \mathcal{W}_2^*$. Let $t \in I_T$. By Theorem 4.4 of Dudley and Norvaiša (1999) and by Proposition 6 of Norvaiša (1999), the product integral $\mathcal{P}(\mathcal{S}(f))(t)$ exists and has the representation

$$\mathcal{P}(\mathcal{S}(f))(t) = \frac{f(t)}{f(0)}\exp\Big\{-\sum_{(0,t]}\Big[\log\Big(\frac{f}{f_-}\Big) - \frac{\Delta^- f}{f_-}\Big] - \sum_{[0,t)}\Big[\log\Big(\frac{f_+}{f}\Big) - \frac{\Delta^+ f}{f}\Big]\Big\}\times$$

$$\times\prod_{(0,t]}\Big[(1+\frac{\Delta^- f}{f_-})\exp(-\frac{\Delta^- f}{f_-})\Big]\prod_{[0,t)}\Big[(1+\frac{\Delta^+ f}{f})\exp(-\frac{\Delta^+ f}{f})\Big] = \frac{f(t)}{f(0)},$$

where the last equality follows using the limiting argument as in the part I. The proof of Theorem 2.7 is complete. $\square$

By Propositions 2.3, 2.6 and Theorem 2.7, for $\mathfrak{Q} = \mathfrak{Q}([0,T])$ and for each $t \in I_T$, it follows that

$$\mathcal{E}_{\mathfrak{Q}}(\mathcal{L}_{\mathfrak{Q}}(f))(t) = f(t)/f(0) \quad \text{and} \quad \mathcal{L}_{\mathfrak{Q}}(\mathcal{E}_{\mathfrak{Q}}(g))(t) = g(t) - g(0) \tag{2.13}$$

whenever $f, g \in \mathcal{W}_2^*(I_T)$ are either right– or left–continuous at each point, f is bounded away from zero and all jumps of g are bigger than -1. Next we partly extend the duality between $\mathcal{E}_\lambda$ and $\mathcal{L}_\lambda$ for $\lambda \in \mathfrak{Q}([0,T])$ and for certain functions outside of the class $\mathcal{W}_2^*$.

Proposition 2.8. *Let $g \in \mathcal{W}_p([0,T])$, $1 \leq p < 3$, be continuous and let $\lambda = \{\lambda(m)\colon m \geq 1\}$ be a sequence of partitions $\lambda(m) = \{0 = t_0^m < \cdots < t_{n(m)}^m = T\}$ such that the mesh $|\lambda(m)| \to 0$ with $m \to \infty$.*

(I) *For each* $t \in [0,T]$, *the limit*

$$b_\lambda(g)(t) := \lim_{m\to\infty} \sum_{i=1}^{n(m)} [g(t_i^m \wedge t) - g(t_{i-1}^m \wedge t)]^2 \tag{2.14}$$

exists if and only if (2.9) *so does, and then*

$$\mathcal{E}_\lambda(g)(t) = \exp\{g(t) - g(0) - 2^{-1} b_\lambda(g)(t)\}. \tag{2.15}$$

(II) *Suppose* $b_\lambda(g)$ *from statement* (I) *is defined and continuous on* $[0,T]$. *Then, for each* $t \in [0,T]$, *the limit* (2.4) *exists for* $f = \mathcal{E}_\lambda(g)$, *and satisfies the relation*

$$\mathcal{L}_\lambda(\mathcal{E}_\lambda(g))(t) = g(t) - g(0). \tag{2.16}$$

Proof. To prove statement (I), let $t \in (0,T]$. For each $m \geq 1$, let $l(m) \in \{1,\dots,n(m)\}$ be an integer such that $t \in (t_{l(m)-1}^m, t_{l(m)}^m]$ and let $u_i^m := g(t_i^m \wedge t) - g(t_{i-1}^m)$ for $i = 1,\dots,l(m)$. Since g is continuous, there exists an integer M such that $\max_i |u_i^m| \leq 1/2$ for $m \geq M$. By the Taylor series expansion with remainder, we have $\log(1+u) = u - u^2/2 + 3\theta u^3$ for $|u| \leq 1/2$, where $|\theta| = |\theta(u)| \leq 1$. Then we get the bound

$$\Big|\log\Big(\prod_{i=1}^{l(m)}(1+u_i^m)\Big) - [g(t) - g(0) - \frac{1}{2} s_2(g;\lambda(m))]\Big|$$

$$\leq \sum_{i=1}^{l(m)} \big|\log(1+u_i^m) - u_i^m + \frac{1}{2}(u_i^m)^2\big| \leq 3\sum_{i=1}^{l(m)} |u_i^m|^3 \leq 3 v_p(g) \max_i |u_i^m|^{3-p} \tag{2.17}$$

for all $m \geq M$. This yields statement (I) because g is continuous and $g \in \mathcal{W}_p([0,T])$.

To prove statement (II), suppose that $b := b_\lambda(g)$ is defined and continuous on $[0,T]$. Given $t \in (0,T]$, for each $m \geq 1$, let $l(m)$ be as before, $P := \mathcal{E}_\lambda(g)$ and let $v_i^m := [P(t_i^m \wedge t) - P(t_{i-1}^m)]/P(t_{i-1}^m)$ for $i = 1,\dots,l(m)$. Since g and b are continuous, there exists an integer M such that $\max_i |v_i^m| \leq 1/2$ for $m \geq M$. As in (2.17), since $|e^x - 1| \leq |x| e^{|x|}$ for $x \in \mathbb{R}$, we get

$$\Big|\log P(t) - \sum_{i=1}^{l(m)} v_i^m + \frac{1}{2}\sum_{i=1}^{l(m)} [v_i^m]^2\Big| \leq 3\sum_{i=1}^{l(m)} |v_i^m|^3$$

$$\leq C v_p(g;[0,T]) \max_i |g(t_i^m \wedge t) - g(t_{i-1}^m)|^{3-p} + C b(T) \max_i |b(t_i^m \wedge t) - b(t_{i-1}^m)|^2$$

for some constant C and all $m \geq M$. Since the right side tends to zero with $m \to \infty$, the limit (2.4) exists for $f = P$ because

$$\lim_{m\to\infty} \sum_{i=1}^{l(m)} [v_i^m]^2 = \lim_{m\to\infty} \sum_{i=1}^{l(m)} \Big[g(t_i^m \wedge t) - g(t_{i-1}^m) - \frac{1}{2}[b(t_i^m \wedge t) - b(t_{i-1}^m)]\Big]^2 = b(t).$$

It then follows that

$$\mathcal{L}_\lambda\big(\mathcal{E}_\lambda(g)\big)(t) = \lim_{m\to\infty}\sum_{i=1}^{l(m)} v_i^m = \log P(t) + \frac{1}{2}\lim_{m\to\infty}\sum_{i=1}^{l(m)}[v_i^m]^2 = g(t) - g(0).$$

The proof of Proposition 2.8 is complete. $\square$

2.4. Price and return. We define a price and its return in a duality under minimal restrictions on stochastic processes. To begin with we define a random moment τ_P which can be interpreted as the time of the crash of a stock. In the case when a stock price P is the solution to the Doléans–Dade equation (1.1), τ_P is the first moment t when $P(t) \le 0$. Given a stochastic process $X = \{X(t)\colon t \in [0,T]\}$, let

$$\tau_P = \tau_P(X) := \begin{cases} t, & \text{if } \inf_{s\in[0,t)} X(s) > 0 \text{ and } X(t) \le 0 \text{ for } t \in (0,T], \\ t+, & \text{if } \inf_{s\in[0,t]} X(s) > 0 \text{ and } X^-(t+) \le 0 \text{ for } t \in (0,T), \\ T+, & \text{if } \inf_{s\in[0,T]} X(s) > 0, \end{cases} \tag{2.18}$$

where $f^-(t+) := \limsup_{s\downarrow t} f(s)$. Let K be the set consisting of points 0, $0+$, $t-$, t, $t+$ for $t \in (0,T]$ with the natural linear ordering: $s+ < t- < t < t+$ if $s < t$, and endowed with the interval topology. Then τ_P is the random variable with values in K. If X is a price, then the event $\{\tau_P = t\}$ can be interpreted as the crash right before the time t, while the event $\{\tau_P = t+\}$ can be interpreted as the crash right after the time t. Also, let $[0,t+) := [0,t]$.

Recalling notation $\mathfrak{Q}([0,T])$, $\mathcal{L}_\lambda$ and $\mathcal{E}_\lambda$ from Definitions 2.2 and 2.5, we have:

Definition 2.9. Let $R = \{R(t)\colon t \in [0,T]\}$ and $P = \{P(t)\colon t \in [0,T]\}$ be stochastic processes on a complete probability space $(\Omega, \mathcal{F}, \Pr)$ such that $R(0) = 0$ and $P(0) = 1$ almost surely.

(1) The pair (P,R) will be called the *weak evolutionary system* on $[0,\tau_P)$ if, for each $\lambda \in \mathfrak{Q}([0,T])$, there exists $N = N(\lambda) \in \mathcal{F}$ with $\Pr(N) = 0$ such that, for each $\omega \in \Omega\setminus N$, the functions $\mathcal{E}_\lambda(R(\cdot,\omega))$, $\mathcal{L}_\lambda(P(\cdot,\omega))$ are defined on $[0,T]\cap[0,\tau_P(P(\omega)))$, and satisfy the relations

$$P(t,\omega) = \mathcal{E}_\lambda(R(\cdot,\omega))(t) \quad \text{and} \quad R(t,\omega) = \mathcal{L}_\lambda(P(\cdot,\omega))(t)$$

for each $t \in [0,T]\cap[0,\tau_P(P(\omega)))$.

(2) If, in addition, the null set $N \in \mathcal{F}$ in (1) can be chosen the same for all $\lambda \in \mathfrak{Q}([0,T])$, then the pair (P,R) will be called the *evolutionary system* on $[0,\tau_P)$.

If the pair (P,R) is the weak evolutionary system then we call P the price and R the return.

We show that if R is a Brownian motion and P is a geometric Brownian motion, then the pair (P,R) is the weak evolutionary system but not the evolutionary system (Proposition 2.11 and Remark 2.12). However, according to the following statement, if almost all sample functions of R are in $\mathcal{W}_2^*([0,T])\cap\mathcal{D}([0,T])$ then (P,R) is the evolutionary system. Recall that $f \in \mathcal{D}([0,T])$ if, at each point of $(0,T)$, f is is either right–continuous or left–continuous.

Next we define a random moment τ_R for a return. Given a stochastic process $Y = \{Y(t)\colon t \in [0,T]\}$ with almost all sample functions in $\mathcal{D}([0,T])$, let

$$\tau_R = \tau_R(Y) := \begin{cases} t, & \text{if } \inf_{s\in[0,t)} \Delta Y(s) > -1 \text{ and } \Delta^- Y(t) \le -1 \text{ for } t \in (0,T], \\ t+, & \text{if } \inf_{s\in[0,t)} \Delta Y(s) > -1 \text{ and } \Delta^+ Y(s) \le -1 \text{ for } t \in (0,T), \\ T+, & \text{if } \inf_{s\in[0,T]} \Delta Y(s) > -1. \end{cases}$$

Here $\Delta Y(s) := 0$ everywhere except at jump points s of Y where either $\Delta Y(s) := \Delta^+ Y(s)$ if it is non-zero, or $\Delta Y(s) := \Delta^- Y(s)$ if it is non-zero.

Proposition 2.10. *Let $R = \{R(t)\colon t \in [0,T]\}$ be a stochastic process on a complete probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$ such that $R(0) = 0$ and $R \in \mathcal{W}_2^*([0,T]) \cap \mathcal{D}([0,T])$ with probability 1. Then the indefinite product integral $\mathcal{P}(R)$ is defined with respect to almost every sample function of R, $\tau_P(\mathcal{P}(R)) = \tau_R(R)$ almost surely and the pair $(\mathcal{P}(R), R)$ is the evolutionary system on $[0, \tau_R)$.*

Proof. Let $N \in \mathcal{F}$ be such that $\mathrm{Pr}(N) = 0$ and $R(\cdot,\omega) \in \mathcal{W}_2^*([0,T]) \cap \mathcal{D}([0,T])$ for all $\omega \in \Omega \setminus N$. Let $\lambda = \{\lambda(m)\colon m \ge 1\} \in \mathfrak{Q}([0,T])$. For each $t < \tau(\omega) := \tau_R(R(\omega))$, let $P(t,\omega) := \mathcal{E}_\lambda(R(\cdot,\omega))(t)$ if $\omega \in \Omega \setminus N$ and $P(t,\omega) := 0$ if $\omega \in N$. By Proposition 2.6, $P(t,\omega) = \prod_0^t (1 + dR(\cdot,\omega))$ for $t < \tau(\omega)$ and $\omega \in \Omega \setminus N$. For all $\omega \in \Omega$ and $t \ge \tau(\omega)$, let $P(t,\omega) := \prod_0^{\tau(\omega)} (1 + dR(\cdot,\omega))$. Then $P = \{P(t)\colon t \in [0,T]\}$ is a stochastic process by construction. Fix $\omega \in \Omega \setminus N$ and let $\tau := \tau(\omega)$, $P(t) := P(t,\omega)$, $R(t) := R(t,\omega)$. Then $P(\tau-)$ exists and $\inf\{P(t)\colon t \in [0,\tau)\} \ge \delta$ for some $\delta > 0$ by Proposition 4.30 of Dudley and Norvaiša (1999). By Lemmas 5.1 and 5.2 of Dudley and Norvaiša (1999), P and R have the same jump points on $[0,\tau)$ and thus $P \in \mathcal{D}([0,\tau))$. Moreover, $P \in \mathcal{W}_2^*([0,\tau))$ by Theorem 2.7. By Proposition 2.3, $\widetilde{R}(t) := \mathcal{L}_\lambda(P)(t)$ is defined for each $t < \tau$ and has the same value with the indefinite LY integral $\mathcal{S}(P)(t)$. By Theorem 2.7 again, $\widetilde{R}(t) = (LY) \int_0^t P^{-1}\, dP = R(t)$ for each $t < \tau$. Since the null set N does not depend on λ, the pair (P, R) is evolutionary system on $[0, \tau_P)$. It is clear that $\tau_P(P(\omega)) = \tau_R(R(\omega))$ for all $\omega \in \Omega \setminus N$. The proof of Proposition 2.10 is complete. $\square$

For example, a fractional Brownian motion B_H with $H \in (1/2, 1)$ and a symmetric α-stable Lévy motion X_α with $\alpha \in (0,2)$ are the returns of the evolutionary systems (P_H, B_H) and (P_α, X_α), respectively, where

$$P_H(t) := \exp\{B_H(t)\} \quad \text{and} \quad P_\alpha(t) := \exp\{X_\alpha(t)\} \prod_{(0,t]} (1 + \Delta^- X_\alpha) \exp\{-\Delta^- X_\alpha\}$$

for $t \in [0,T]$. The price P_α is positive until the first moment t when $\Delta^- X_\alpha(t) \le -1$.

Proposition 2.11. *Let $B = \{B(t)\colon t \in [0,T]\}$ be a standard Brownian motion and let $P_B := \{\exp\{B(t) - t/2\}\colon t \in [0,T]\}$. Then the pair (P_B, B) is the weak evolutionary system on $[0,T]$.*

Remark 2.12. The proof of the above proposition rely on Théorème 5 of Lévy (1940, p. 510): for a standard Brownian motion B and for a sequence $\lambda = \{\lambda(m)\colon m \ge 1\} \in \mathfrak{Q}([0,1])$, the limit $\lim_{m\to\infty} s_2(B; \lambda(m)) = 1$ exists with probability 1. However, the exceptional null set $N(\lambda) \in \mathcal{F}$ of this implication depends on λ and $\cup\{N(\lambda)\colon \lambda \in$

$\mathfrak{Q}([0,1])\} = \Omega$. Moreover, for almost all $\omega \in \Omega$ there exist $\lambda \in \mathfrak{Q}([0,1])$ such that $\lim_{m\to\infty} s_2(B(\cdot,\omega);\lambda(m)) = \infty$, and hence, $\mathcal{E}_\lambda(B(\cdot,\omega)) = 0$. The proofs of these properties are given by Freedman (1983, p. 48) because $\cup_m \lambda(m)$ is everywhere dense in $[0,1]$ if and only if the mesh $|\lambda(m)| \to 0$ with $m \to \infty$.

Proof. The claim will follow from Proposition 2.8 once we show that, given $\lambda = \{\lambda(m)\colon m \geq 1\} \in \mathfrak{Q}([0,T])$, the limit (2.14) with $g = B$ exists with probability 1 for each $t \in [0,T]$. For each $t \in (0,T]$ and $m \geq 1$, let $b_m(t,\omega) := \sum_{i=1}^{n(m)} \big[B(t_i^m \wedge t,\omega) - B(t_{i-1}^m \wedge t,\omega)\big]^2$. Let $N = N(\lambda) \in \mathcal{F}$ be a null set such that, for each $\omega \in \Omega \setminus N$, $B(\cdot,\omega)$ is continuous function of bounded p-variation for some $2 < p < 3$, and the limit $\lim_m b_m(t,\omega) = t$ exists for each t in the countable set $S := \cup_m \lambda(m)$. Let $t \in (0,T) \setminus S$. Since $b_m(t,\omega) - t = [B(t,\omega) - B(t_{i-1}^m,\omega)]^2 + b_m(t_{i-1}^m,\omega) - t_{i-1}^m + [t_{i-1}^m - t]$ for $t_{i-1}^m < t < t_i^m$, it follows that the limit as $m \to \infty$ of $b_m(t,\omega)$ is t for each $t \in [0,T]$. Therefore an appeal to Proposition 2.8 completes the proof. $\square$

3 Pathwise trading strategies

A trading strategy is a collection of instructions for buying and selling a stock, depending on its price fluctuations. A mathematical notion of a trading strategy should be defined so that one can calculate the portfolio value and portfolio gain for each single trajectory of a stock price. In the stochastic exponential model, the portfolio gain is the stochastic integral of a trading strategy with respect to a price. Its value can be approximated by portfolio gains based on simple trading strategies in probability. In this section we take a pathwise approach to trading strategies.

Consider a frictionless stock market with $\nu+1$ non-dividend-paying stocks and open for trade during a time period $[0,T]$. A vector function $P = (P_0,\ldots,P_\nu)$ defined on $[0,T]$ will be called the *price* during the time period $[0,T]$ if, for each $k = 0,\ldots,\nu$, $\inf\{P_k(t)\colon t \in [0,T]\} > 0$. The value $P_k(t)$ refers to the price of the kth stock at time $t \in [0,T]$ for $k = 0,\ldots,\nu$. For example, the price may be a vector of sample functions of the price stochastic processes defined in the preceding section. A possible dependence of the price on a randomness and the duality relation between the price and its return are disregarded in this section.

As before, $\mathfrak{Q}([0,T])$ denotes the set of all nested sequences $\lambda = \{\lambda(m)\colon m \geq 1\}$ of partitions of $[0,T]$ such that $\cup_m \lambda(m)$ is dense in $[0,T]$. A sequence $\kappa = \{\kappa(m)\colon m \geq 1\} \in \mathfrak{Q}([0,T])$ is a *refinement* of a sequence $\lambda = \{\lambda(m)\colon m \geq 1\} \in \mathfrak{Q}([0,T])$ if each $\kappa(m)$ is a refinement of $\lambda(m)$.

Definition 3.1. Let $P = (P_0,\ldots,P_\nu)$ be a price during a time period $[0,T]$. Given $\lambda \in \mathfrak{Q}([0,T])$, a vector function $\phi = (\phi_0,\ldots,\phi_\nu)$ defined on $[0,T]$ will be called the (λ,P)*–trading strategy* during the time period $[0,T]$ if, for each $k = 0,\ldots,\nu$ and $t \in [0,T]$, there exists the limit

$$(LCS)\int_0^t \phi_k\, d_\lambda P_k := \lim_{m\to\infty} \sum_{i=1}^{n(m)} \phi_k(t_{i-1}^m \wedge t)[P_k(t_i^m \wedge t) - P_k(t_{i-1}^m \wedge t)], \tag{3.1}$$

where $\lambda = \{\lambda(m)\colon m \geq 1\}$ and $\lambda(m) = \{t_i^m\colon i = 0,\ldots,n(m)\}$. If there exists $\lambda_0 \in \mathfrak{Q}([0,T])$ such that for each refinement $\lambda \in \mathfrak{Q}([0,T])$ of λ_0, ϕ is the (λ,P)–trading

strategy on $[0,T]$ and (3.1) does not depend on λ, then we call ϕ the *P–trading strategy* during the time period $[0,T]$ and replace d_λ with d in the left side of (3.1).

Föllmer (1981) proved that (3.1) exists whenever $\phi_k = f{\circ}P_k$ for some $f \in C^1$ and the quadratic variation is defined for P_k along the sequence $\lambda \in \mathfrak{Q}([0,T])$. The above notion of (λ, P)–trading strategy is similar to the convergence of trading strategies introduced by Bick and Willinger (1994, p. 356). These authors derived the Black and Scholes formula without probabilistic arguments using Föllmer's variant of Itô's formula. Remark 2.12 ensure that ϕ may be (λ, P)–trading strategy without being a P-trading strategy. It is clear that each ϕ_k is Left Cauchy–Stieltjes integrable with respect to P_k if ϕ is P–trading strategy. As the rest of this section show we could use the LY integral instead of the LCS integral in the definition of P–trading strategies. However in econometric analysis it seems easier to handle with the latter integral because to evaluate the LY integral we would need to know jumps of the price. The following statement provides sufficient conditions for a vector function to be the P–trading strategy.

Proposition 3.2. *Let $P = (P_0, \ldots, P_\nu)$ be a price during a time period $[0,T]$. A vector function $\phi = (\phi_0, \ldots, \phi_\nu)$ on $[0,T]$ is the P–trading strategy, each ϕ_k is LY integrable with respect to P_k on $[0,T]$ and*

$$(LCS)\int_0^t \phi_k\,dP_k = (LY)\int_0^t \phi_k\,dP_k \tag{3.2}$$

for each $t \in [0,T]$ and each $k = 0, \ldots, \nu$ in either of the following two cases:

1. *for each $k = 0, \ldots, \nu$, $P_k \in \mathcal{W}_2^*([0,T])$ and $\phi_k = f_k{\circ}P_k$ for some $f_k\colon \mathbb{R} \mapsto \mathbb{R}$ satisfying the local Lipschitz condition.*
2. *for each $k = 0, \ldots, \nu$, $P_k \in \mathcal{W}_p([0,T])$ and $\phi_k \in \mathcal{W}_q([0,T])$ with $p, q > 0$, $1/p + 1/q > 1$.*

Proof. In case (2) the conclusion follows from Theorem 2 and Corollary 3 of Norvaiša (1999). To prove the conclusion in case (1), for notation simplicity we suppress the index $k = 0, \ldots, \nu$ for f_k and P_k. By Theorem 2.1, $f{\circ}P$ is LY integrable with respect to P on $[0,T]$ and

$$(LY)\int_0^t (f{\circ}P)\,dP = F{\circ}P(t) - F{\circ}P(0) - \sum_{(0,t]}\phi^- - \sum_{[0,t)}\phi^+,$$

where $\phi^- := \Delta^-(F{\circ}P) - (f{\circ}P)_-\Delta^-P$, $\phi^+ := \Delta^+(F{\circ}P) - (f{\circ}P)\Delta^+P$ and $F(u) := \int_0^u f(x)\,dx$ for $u \geq 0$. Fix $t \in (0,T]$ and let $\epsilon > 0$. By Lemma 2.4, there exists $\lambda = \{s_j\colon j = 0, \ldots, m\} \in Q([0,t])$ such that $\sum_{j=1}^m v_2(P; (s_{j-1}, s_j)) < \epsilon$, and for each refinement $\{t_i\colon i = 0, \ldots, n\}$ of λ,

$$\Big|\sum_{(0,t]}\phi^- + \sum_{[0,t)}\phi^+ - \sum_{i=1}^n \big[\phi^-(t_i) + \phi^+(t_{i-1})\big]\Big| < \epsilon.$$

Then choose $\{u_{j-1}, v_j\colon j = 1, \ldots, m\} \subset [0,t]$ such that $s_{j-1} < u_{j-1} < v_j < s_j$ for $j = 1, \ldots, m$ and

$$\sum_{j=1}^m \Big[Osc\,(P; (z_{j-1}, u_{j-1}]) + Osc\,(P; [v_j, z_j))\Big] < \epsilon.$$

Then by the mean value theorem and using the Lipschitz condition with the constant K, we have

$$\Big|S_{LCS}(f\circ P, P;\kappa) - \big[F\circ P(t) - F\circ P(0) - \sum_{(0,t]}\phi^- - \sum_{[0,t)}\phi^+\big]\Big|$$

$$< \epsilon + \sum_{i=1}^{n}\Big|f\circ P(t_{i-1})[P(t_i)-P(t_{i-1})] - [F\circ P(t_i) - F\circ P(t_{i-1})] - [\phi^-(t_i) - \phi^+(t_{i-1})]\Big|$$

$$= \epsilon + \sum_{i=1}^{n}\Big|[f(\theta_i) - f(P(t_i-))][P(t_i-) - P(t_{i-1}+)] + [f\circ P(t_i-) - f\circ P(t_{i-1})]$$

$$\times[P(t_i) - P(t_{i-1}+)]\Big| < \epsilon + K\epsilon + 2\sup_t |P(t)|K\epsilon + K\epsilon,$$

where $\theta_i \in [P(t_{i-1}+)\wedge P(t_i-), P(t_{i-1}+)\vee P(t_i-)]$. Since ϵ is arbitrary the proof of Proposition 3.2 is complete. $\square$

Having defined the P–trading strategies via the Left Cauchy–Stieltjes integral, the following definition of self–financing strategy corresponds naturally to its counterpart in the stochastic exponent model.

Definition 3.3. Let $P = (P_0,\dots,P_\nu)$ be a price during a time period $[0,T]$ and let $\phi = (\phi_0,\dots,\phi_\nu)$ be the P–trading strategy.

(1) The real-valued functions $V^\phi = V^{\phi,P}$ and $G^\phi = G^{\phi,P}$ defined on $[0,T]$ by

$$V^\phi(t) := \sum_{k=0}^{\nu}\phi_k(t)P_k(t) \quad \text{and} \quad G^\phi(t) := \sum_{k=0}^{\nu}(LCS)\int_0^t \phi_k\,dP_k$$

are called the *portfolio value function* and the *portfolio gain function*, respectively.

(2) The P–trading strategy ϕ is called *self–financing* if $V^\phi(t) = V^\phi(0) + G^\phi(t)$ for each $t\in[0,T]$.

For the sake of illustration, next we partially extend Proposition 3.24 of Harrison and Pliska (1981) to the present setting. Let us denote the discounted price $(1, P_1/P_0,\dots,P_\nu/P_0)$ by $\overline{P}$.

Proposition 3.4. *Let $0<p<2$ and let $P=(P_0,\dots,P_\nu)$ be a price during a time period $[0,T]$ such that $P_k\in\mathcal{W}_p([0,T])$ for $k=0,\dots,\nu$. Suppose that $\phi=(\phi_0,\dots,\phi_\nu)$ is a vector function on $[0,T]$ such that $\phi_k\in\mathcal{W}_p([0,T])$ for $k=0,\dots,\nu$. Then ϕ is self–financing P–trading strategy if and only if it is self–financing $\overline{P}$–trading strategy.*

Proof. Let $\beta := 1/P_0$. We have $\beta P_k\in\mathcal{W}_p([0,T])$ for each $k=1,\dots,\nu$. Thus, by case (2) of Proposition 3.2 with $q=p$, ϕ is the P–trading strategy if and only if ϕ is the $\overline{P}$–trading strategy

To prove the "only if" part of the statement suppose ϕ is self–financing P-trading strategy. Let $V := V^{\phi,P}$ and $\overline{V} := V^{\phi,\overline{P}} = \beta V$. By Theorem 2.1 with $F(u) = u_1u_2$ for $u = (u_1, u_2)$, $h = \phi_k$, and $\alpha = 1$, for each $t \in [0, T]$, we have

$$(LY)\int_0^t \phi_k\, d(\beta P_k) = (LY)\int_0^t \phi_k\beta\, dP_k + (LY)\int_0^t \phi_k P_k\, d\beta + \sum_{(0,t]}(\phi_k)_-\Delta^-\beta\Delta^- P_k \tag{3.3}$$

for $k = 0, \ldots, \nu$, where the left side of (3.3) is equal to 0 when $k = 0$. Since P–trading strategy ϕ is self–financing, by the substitution rule for the Left Young integral (Theorem 9 of Norvaiša, 1999), for each $t \in [0, T]$, we have

$$(LY)\int_0^t \beta\, dV = \sum_{k=0}^{\nu}(LY)\int_0^t \beta\, d\Big((LY)\int_0^{\cdot} \phi_k\, dP_k\Big) = \sum_{k=0}^{\nu}(LY)\int_0^t \beta\phi_k\, dP_k. \tag{3.4}$$

Using Theorem 2.1 again except that now $h = 1$, and Proposition 7 of Norvaiša (1999) about jumps of the indefinite LY integral, for each $t \in [0, T]$, we get

$$\begin{aligned} \overline{V}(t) - \overline{V}(0) &= (LY)\int_0^t \beta\, dV + (LY)\int_0^t V\, d\beta + \sum_{(0,t]} \Delta^-\beta\Delta^- V \\ \text{by (3.4)} \quad &= \sum_{k=0}^{\nu}\Big\{(LY)\int_0^t \phi_k\beta\, dP_k + (LY)\int_0^t \phi_k P_k\, d\beta + \sum_{(0,t]}(\phi_k)_-\Delta^-\beta\Delta^- P_k\Big\} \\ \text{by (3.3)} \quad &= \sum_{k=1}^{\nu}(LY)\int_0^t \phi_k\, d(\beta P_k) = G^{\phi,\overline{P}}(t). \end{aligned}$$

Thus ϕ is self–financing $\overline{P}$–trading strategy. The proof of the converse implication is similar and therefore is omitted. $\square$

We finish with the main argument in favor of the pathwise approach to trading strategies. In their discussion of the notion of trading strategy, Harrison and Pliska (1981, Section 7) made several suggestions. For example, it would be desirable to show that a claim is attainable if and only if it is the limit (in some appropriate sense) of claims generated by simple self–financing strategies. Duffie and Protter (1992) and Eberlein (1992) proved that the portfolio gain processes are approximable by their discrete counterparts under certain conditions. Next we show a kind of approximation of a "contingent claim" by simple self–financing strategies in the present context. A trading strategy $\phi = (\phi_0, \ldots, \phi_\nu)$ is *simple* if each ϕ_k is a step function on $[0, T]$. The idea of the following statement originated from Harrison, Pitbladdo and Schaefer (1984, Proposition 9), where this claim is proved for price processes with continuous sample functions of bounded variation.

Theorem 3.5. *Let P be a price during a time period $[0, T]$ and let ϕ be a vector function on $[0, T]$, both satisfying either of the two conditions of Proposition 3.2. If $\inf\{V^\phi(t) \colon t \in [0, T]\} > 0$ then there exists a sequence $\{\phi^N \colon N \geq 1\}$ of simple self–financing P–trading strategies such that $V^{\phi^N}(0) = V^\phi(0)$ and $\lim_{N\to\infty} V^{\phi^N}(T) = V^\phi(T)$.*

Proof. We start with the construction of the sequence of simple self–financing P–trading strategies based on a given nested sequence $\{\lambda^N \colon N \geq 1\}$ of partitions $\lambda^N =$

$\{0 = t_0 < \cdots < t_n = T\}$. Given an integer $N \geq 1$, we define $\phi^N = (\phi_0^N, \ldots, \phi_\nu^N)$ recursively with constant values on each interval of the partition λ^N. For each $k = 0, \ldots, \nu$, let $\phi_k^N := \phi_k(0)$ on $[0, t_1)$. Suppose that all ϕ_k^N are defined on $[0, t_i)$ for some $1 \leq i \leq n$. Let $V^N(t_i) := \sum_{k=0}^{\nu} \phi_k^N(t_{i-1})P_k(t_i)$. Then for each $k = 0, \ldots, \nu$, let ϕ_k^N be equal to $\phi_k(t_i)V^N(t_i)/V^\phi(t_i)$ either on $[t_i, t_{i+1})$ if $i < n$, or on $\{T\}$ if $i = n$. It is clear that each ϕ^N is a simple P–trading strategy. Moreover, the portfolio value function V^{ϕ^N} has values $V^{\phi^N}(0) = V^\phi(0)$ and

$$V^{\phi^N}(t_i) = \sum_{k=0}^{\nu} \phi_k^N(t_i)P_k(t_i) = \frac{V^N(t_i)}{V^\phi(t_i)} \sum_{k=0}^{\nu} \phi_k(t_i)P_k(t_i) = V^N(t_i) \tag{3.5}$$

for each $i = 1, \ldots, n$. Next we show that each P–trading strategy ϕ^N is self–financing. Let $u = t_{i-1}$ and $v \in (t_{i-1}, t_i]$ for some $i = 1, \ldots, n$. Since ϕ_k^N is constant on $[u, v)$ we get

$$(LY)\int_u^v \phi_k^N \, dP_k = (RS)\int_u^v (\phi_k^N)_-^{(u)} \, d(P_k)_+^{(v)} + \phi_k^N(u)\Delta^+ P_k(u)$$
$$= \phi_k^N(u)[P_k(v) - P_k(u+)] + \phi_k^N(u)\Delta^+ P_k(u) = \phi_k^N(u)[P_k(v) - P_k(u)]$$

for each $k = 0, \ldots, \nu$. Given $t \in (0, T]$, let $l := \max\{i \leq n \colon t_i \leq t\}$. Then using the additivity of the LY integral over adjacent intervals (Theorem 4 of Norvaiša, 1999) and changing the order of summation over k and i, we get

$$\begin{aligned} G^{\phi^N}(t) &= \sum_{k=0}^{\nu} \Big\{ (LY)\int_{t_l}^t \phi_k^N \, dP_k + \sum_{i=1}^{l} (LY)\int_{t_{i-1}}^{t_i} \phi_k^N \, dP_k \Big\} \\ &= \sum_{k=0}^{\nu} \phi_k^N(t_l)[P_k(t) - P_k(t_l)] + \sum_{i=1}^{l} \big[V^N(t_i) - V^{\phi^N}(t_{i-1})\big] \\ \text{by (3.5)} \quad &= \sum_{k=0}^{\nu} \big[\phi_k^N(t_{i(t)})P_k(t) - \phi_k^N(0)P_k(0)\big] = V^{\phi^N}(t) - V^{\phi^N}(0). \end{aligned}$$

Thus the P–trading strategy ϕ^N is self–financing for each $N \geq 1$. By (3.5) and by the additivity of the LY integral over adjacent intervals again, we get

$$\Delta^N(t_{i-1}, t_i) := V^\phi(t_i) - V^\phi(t_{i-1}) \frac{V^{\phi^N}(t_i)}{V^{\phi^N}(t_{i-1})} = V^\phi(t_i) - \sum_{k=0}^{\nu} \phi_k(t_{i-1})P_k(t_i)$$

$$= \sum_{k=0}^{\nu} \Big\{ (LY)\int_{t_{i-1}}^{t_i} \phi_k \, dP_k - \phi_k(t_{i-1})[P_k(t_i) - P_k(t_{i-1})] \Big\} =: \sum_{k=0}^{\nu} \Delta_k^N(t_{i-1}, t_i) \tag{3.6}$$

for each $i = 1, \ldots, n$. Suppose that one can choose a sequence $\{\lambda^N \colon N \geq 1\}$ such that

$$\max_{1 \leq i \leq n} |\Delta^N(t_{i-1}, t_i)| \leq \sum_{i=1}^{n} |\Delta^N(t_{i-1}, t_i)| \leq \epsilon_N \tag{3.7}$$

for some $\epsilon_N \downarrow 0$. Then it follows that

$$\Big|\frac{V^{\phi^N}(t_i)}{V^{\phi}(t_i)}\frac{V^{\phi}(t_{i-1})}{V^{\phi^N}(t_{i-1})} - 1\Big| = \Big|\frac{\Delta^N(t_{i-1},t_i)}{V^{\phi}(t_i)}\Big| \le \epsilon_N/\delta$$

for each $t_{i-1}, t_i \in \lambda^N$. We conclude then recursively that $V^N(t_i) \ne 0$ for $i = 1,\ldots,n$ whenever $2\epsilon_N \le \delta$. By the mean value theorem, $|\log(1+u)| \le 2|u|$ for each $|u| \le 1/2$. Thus, for all N such that $2\epsilon_N \le \delta$, using the telescoping sum representation, we get

$$\begin{aligned}\Big|\log\frac{V^{\phi^N}(T)}{V^{\phi}(T)}\Big| &\le \sum_{i=1}^{n}\Big|\log\frac{V^{\phi^N}(t_i)}{V^{\phi}(t_i)} - \log\frac{V^{\phi^N}(t_{i-1})}{V^{\phi}(t_{i-1})}\Big| \le 2\sum_{i=1}^{n}\Big|\frac{V^{\phi^N}(t_i)}{V^{\phi}(t_i)}\frac{V^{\phi}(t_{i-1})}{V^{\phi^N}(t_{i-1})} - 1\Big| \\ &\le \frac{2}{\delta}\sum_{i=1}^{n}|\Delta^N(t_{i-1},t_i)| \le 2\epsilon_N/\delta,\end{aligned}$$

where the last inequality follows from the second inequality in (3.7). Since $V^{\phi^N}(0) = V^{\phi}(0)$ by construction, this yields the conclusion of the theorem.

It remains to find $\{\lambda^N\colon N \ge 1\}$ such that (3.7) holds for some $\epsilon_N \downarrow 0$. Suppose that condition (1) of Proposition 3.2 holds. By Theorem 2.1 and by the mean value theorem, each term $\Delta_k^N(u,v)$ in (3.6) with $u = t_{i-1}$, $v = t_i$, $i = 1,\ldots,n$ and $k = 0,\ldots,\nu$ is equal to

$$\begin{aligned}&[f_k(\theta_k) - f_k(P_k(v-))][P_k(v-) - P_k(u+)] + [f_k(P_k(v-)) - f_k(P_k(u))][P_k(v) - P(u+)] \\ &\quad - \sum_{(u,v)}\big[\Delta^-(F_k\circ P_k) - (f_k\circ P_k)_-\Delta^- P_k\big] - \sum_{(u,v)}\big[\Delta^+(F_k\circ P_k) - (f_k\circ P_k)\Delta^+ P_k\big],\end{aligned}$$

where $\theta_k \in (P_k(u+)\wedge P_k(v-), P_k(u+)\vee P_k(v-)]$ and $F_k(u) = \int_0^u f_k(x)\,dx$ for $u \ge 0$. Then, given $\epsilon_N \downarrow 0$, one can find λ^N such that (3.7) holds just as in the proof of Proposition 2.3. Finally, suppose that condition (2) of Proposition 3.2 holds. Choose $p' > p$ and $q' > q$ so that $1/p' + 1/q' > 1$. By the Love–Young inequality (p. 256 in Young, 1936), each term $\Delta_k^N(u,v)$ in (3.6) with $u = t_{i-1}$, $v = t_i$, $i = 1,\ldots,n$ and $k = 0,\ldots,\nu$ can be bounded as follows:

$$\begin{aligned}|\Delta_k^N(u,v)| &= \Big|(RS)\int_u^v [(\phi_k)_-^{(u)} - (\phi_k)_-^{(u)}(u)]\,d(P_k)_+^{(v)} + \sum_{(u,v)}\Delta^-\phi_k\Delta^+P_k\Big| \\ &\le K V_{p'}(P_k;[u+,v])V_{q'}(\phi_k;[u,v-]) + \sum_{(u,v)}|\Delta^-\phi_k\Delta^+P_k|\end{aligned}$$

for some finite constant K depending on p' and q' only. Again, given $\epsilon_N \downarrow 0$, one can find λ^N such that (3.7) holds as in the proof of Proposition 2.3. The proof of Theorem 3.5 is complete. □

4 Implications and conclusions

The results of the present paper provide an alternative construction of a stock price model, and show that many concrete financial models can be treated using classical

calculus. By its definition, the evolutionary system is the continuous–time model obtained as the limit of the discrete–time model (2.2) along a sequence of partitions of a time interval into shrinking subintervals. The evolutionary system separates analytical and probabilistic aspects of analysis casting new light on important problems of stock price modelling. The arbitrage construction in Subsection 1.5 illustrates implications of this separation.

New definition of the return R in the evolutionary system (P, R) makes easier to use it in a statistical analysis as compared with the definition (1.2). Statistical analysis of analytical properties of functions developed in relation to natural sciences could be applied in econometric analysis of the evolutionary system. For example, an interesting task is to distinguish the hypotheses that the p-variation index $\upsilon(R) < 2$ against the hypotheses that $\upsilon(R) \geq 2$. This is important because the value $\upsilon(R) = 2$ separates a fundamentally different behaviour of R. Also, testing hypotheses $\upsilon(R) < 2$ and R is continuous could be used to test market efficiency related to arbitrage. Naturally that there are no ready to use statistical tests for estimating the p-variation index. In this case one needs to extract from data an information about a local behavior of a sample function rather than an information about tail distribution, or correlation estimates. The first step in this direction has been taken up by Norvaiša and Salopek (1999). These authors suggest a statistic based on old results of G. Baxter and E.G. Gladyshev concerning quadratic variation for Gaussian processes. Also, they compare the results of data analysis using the new definition of the return and the log return.

Let $\dim_{HB}(G)$ be the Hausdorff–Besicovitch dimension of a set G. Then for a large class of stochastic processes, the relation $\dim_{HB}(\text{graph } X) = 2 - 1/(1 \vee \upsilon(X))$ holds for almost all sample functions of X. This fact can be used to construct new statistics for estimating the p-variation index $\upsilon(X)$ because statistical analysis of fractal dimensions is already highly developed in various natural sciences. The real analysis approach to modelling of stock price changes provides a new meaning to stylized facts discovered in Econophysics (see e.g. Bouchaud and Potters, 1999), and opens a way for exploring new tools for investigating financial markets.

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